Mathematics 221 **Spring 2004** **Project 10**

Due April 9, 2004

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Collaborators **Name**

Directions: Be sure to follow the guidelines for writing up projects as specified in the course information sheet (passed out on the first day of class). Whenever appropriate, use in-line citations, including page numbers and people consulted when you present information obtained from discussion, a text, notes, or technology. **Only write on one side of each page.**

"No, no, you're not thinking, you're just being logical." -Niels Bohr, physicist (1885-1962)

Project Description

In this project we will be finding the "volume" of spherical balls in various dimensions.

We are all familiar with the idea of a spherical ball in dimension 3. Basketballs, softballs, soccer balls are all good images to keep in mind. However, the mathematical definition of a 3 - dimensional ball (of radius R) is the set

$$B^3 = \left\{ (x, y, z) : x^2 + y^2 + z^2 \leq R^2 \right\}.$$

In a similar fashion we can define balls for all of the other positive dimensions as follows.

$$\begin{aligned}
B^1 &= \left\{ x : \ x^2 \leq R^2 \right\} \\
B^2 &= \left\{ (x, y) : \ x^2 + y^2 \leq R^2 \right\} \\
B^3 &= \left\{ (x, y, z) : \ x^2 + y^2 + z^2 \leq R^2 \right\} \\
B^4 &= \left\{ (x_1, x_2, x_3, x_4) : \ x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq R^2 \right\} \\
&\vdots \\
B^n &= \left\{ (x_1, x_2, \cdots, x_n) : \ x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2 \right\}
\end{aligned}$$

Now, let V_n denote the volume of the n - dimensional ball B^n and use polar coordinates in the first two coordinate positions for points in $\mathbf{R}^n$. That is, each point of $\mathbf{R}^n$ can be written in the form $(r, \theta, x_3, \cdots x_n)$ where $r = \sqrt{x_1^2 + x_2^2}$ and $\tan(\theta) = \frac{x_2}{x_1}$.

Here is another description of B^n.

$$\begin{aligned}
B^n &= \left\{ (x_1, x_2, \cdots, x_n) : \ x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2 \right\} \\
&= \left\{ (r, \theta, \cdots, x_n) : \ r^2 + x_3^2 + \cdots + x_n^2 \leq R^2 \ , \ 0 \leq r \leq R, \ 0 \leq \theta \leq 2\pi \right\} \\
&= \left\{ (r, \theta, \cdots, x_n) : \ x_3^2 + \cdots + x_n^2 \leq R^2 - r^2 \ , \ 0 \leq r \leq R, \ 0 \leq \theta \leq 2\pi \right\}
\end{aligned}$$

This tells us that B^n is the union all of the $(n-2)$- dimensional balls of radius $\sqrt{R^2 - r^2}$ that we get as we range r from 0 to R and θ from 0 to 2π. It is also easy to see that if V_{n-2} is the volume of the $(n-2)$ - dimensional ball of radius R then the volume of the $(n-2)$ - dimensional ball of radius $\sqrt{R^2 - r^2}$ is

$$\left(\frac{\sqrt{R^2 - r^2}}{R} \right)^{n-2} V_{n-2}.$$

1. Write an iterated double integral, I, in polar coordinates that gives the volume V_n of the n - dimensional ball of radius R in terms of the volume of the $(n-2)$ - dimensional ball of radius R.

2. Write out the numerical values for V_1 and V_2 and use these and your formula from part 1. to compute V_3, V_4, V_5 and V_6.